Meet #2
December 2006

Intermediate
Mathematics League
of
Eastern Massachusetts

Meet #2
December 2006

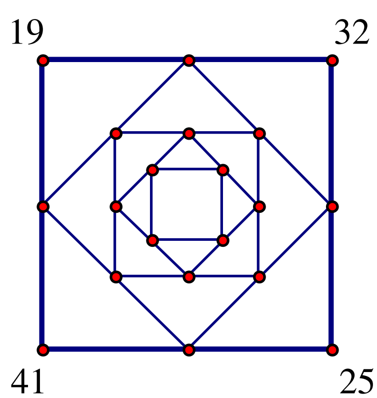
Category 1
Mystery
Meet #2, December 2006

1. If the pattern below is extended to the sixth row, what number would appear in the circle?

$$1^2 + 2^2 + 2^2 = 9 = 3^2$$
$$2^2 + 3^2 + 6^2 = 49 = 7^2$$
$$3^2 + 4^2 + 12^2 = 169 = 13^2$$
$$4^2 + 5^2 + 20^2 = \underline{\qquad} = \underline{\qquad}$$
$$\underline{\qquad\qquad\qquad} = \underline{\qquad} = \underline{\qquad}$$
$$\underline{\qquad\qquad\qquad} = \underline{\qquad} = \underline{\qquad}$$

2. Sadie is 3 years younger than Beth.
Leila is 2 years older than James.
Nathaniel is 4 years older than Sadie.
Leila is 4 years younger than Beth.
How many years older is Nathaniel than Beth?

3. Find the differences for the pairs of numbers on the corners and write the answer on the midpoint between them. Repeat the process for each square. What is the sum of the numbers at the four corners of the inner-most square?



Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 1
Mystery
Meet #2, December 2006

Answers

1. 1849

2. 1

3. 0

1. In the third position of the sums, the base increases by 4, then 6, then 8, etc. The same is true of the base at the far right. The desired value is 43^2 = **1849**.

$$1^2 + 2^2 + 2^2 = 9 = 3^2$$
$$2^2 + 3^2 + 6^2 = 49 = 7^2$$
$$3^2 + 4^2 + 12^2 = 169 = 13^2$$
$$4^2 + 5^2 + 20^2 = 441 = 21^2$$
$$5^2 + 6^2 + 30^2 = 961 = 31^2$$
$$6^2 + 7^2 + 42^2 = \mathbf{1849} = 43^2$$

2. We only need the first and third statements to solve this problem. If Sadie is 3 years younger than Beth, then Beth is 3 years older than Sadie. Nate is 4 years older than Sadie, so Nate must be 4 – 3 = **1** year older than Beth.

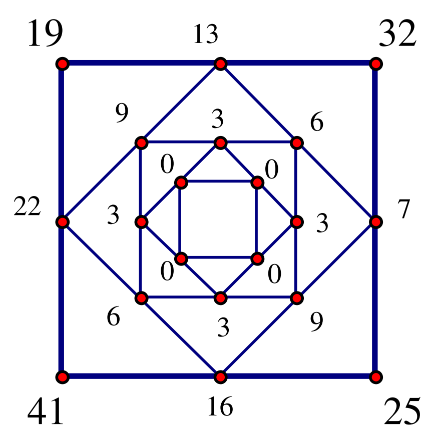
Students with correct answer in a cluster of 6 schools:

1. 28/36

2. 34/36

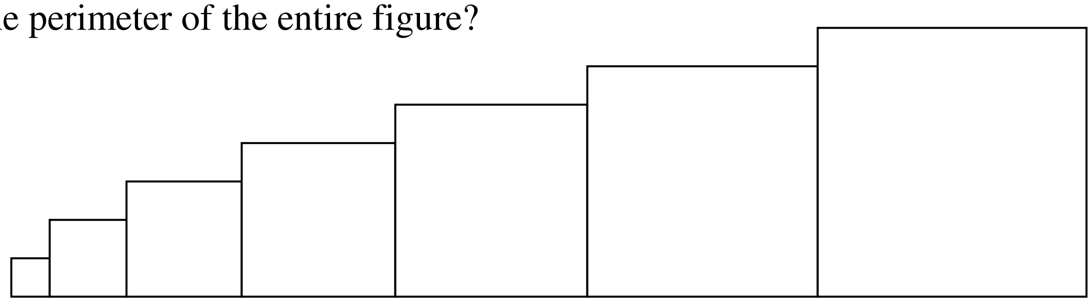
3. 31/36

3. The four numbers at the corners of the inner-most square are all zeros. The desired sum is thus **0**.

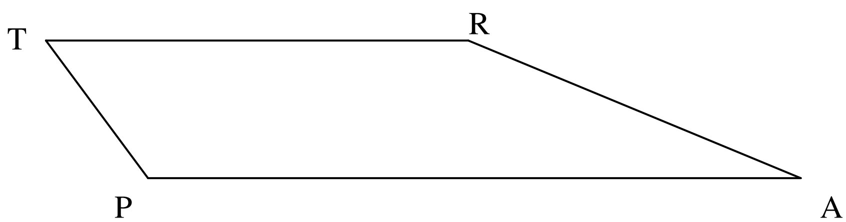


Category 2
Geometry
Meet #2, December 2006

1. The square at the far left in the figure below is a unit square. As you move to the right, the side length of each square increases by 1 unit. How many units are in the perimeter of the entire figure?

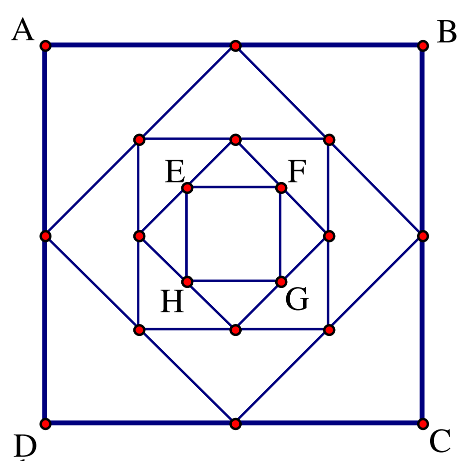


2. In the figure below, sides TR and AP are parallel, but sides TP and RA are not parallel. The length of side TR is 11 units and the length of side AP is 17 units. If the area of TRAP is 84 square units, how many units apart are the two parallel lines?



3. If the area of the inner-most square EFGH is 9 square units, how many units are in the perimeter of the outer-most square ABCD?

Answers
1. _______________
2. _______________
3. _______________



Solutions to Category 2
Geometry
Meet #2, December 2006

Answers

1. 70

2. 6

3. 48

Students with correct answer in a cluster of 6 schools:

1. 28/36

2. 28/36

3. 20/36!
(Many got 144 for #3, which is the AREA of the outer square, not the perimeter.)

1. The distance across the bottom of the figure is 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 units. The sum of the horizontal distances across the tops of the squares is also 28 units. The height of the square at the far right is 7 units. There are also seven 1 unit lengths on the left of each square. The total perimeter of the figure is thus 28 + 28 + 7 + 7 = **70** units.

2. Quadrilateral TRAP is a trapezoid. The formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the two parallel bases. We know the area and the lengths of the two bases. Solving for h in the area formula as shown below, we find that the two parallel lines are **6** units apart.

$$84 = \frac{1}{2}h(11+17)$$
$$84 = \frac{1}{2}h(28)$$
$$84 = 14h$$
$$h = \frac{84}{14} = \mathbf{6}$$

3. Each larger square in the figure is twice the area of the one inside it. Since the area of square EFGH is 9 square units, the area of square ABCD must be $9 \times 2 \times 2 \times 2 \times 2 \times 2 = 144$ square units. The side length of square ABCD must be 12 units, since $12 \times 12 = 144$. The perimeter of square ABCD is thus $4 \times 12 =$ **48** units.

Category 3
Number Theory
Meet #2, December 2006

1. Give the prime factorization of 792. You may use exponents or not, but you must list the primes from least to greatest. For example, the prime factorization of 60 can be written as $2^2 \cdot 3 \cdot 5$ or as $2 \cdot 2 \cdot 3 \cdot 5$.

2. The product of two numbers is 396. If the greatest common factor (GCF) of these two numbers is 3, what is the least common multiple (LCM) of these two numbers?

3. At the manufacturing plant of Gadgets & Gizmos, Inc., it takes 1 hour and 24 minutes to assemble a gadget and 1 hour and 52 minutes to assemble a gizmo. At 9:12 AM one morning, a new gadget and a new gizmo roll off the assembly lines simultaneously. Give the time of day, using AM or PM, for the next time that a new gadget and a new gizmo will roll off the assembly line simultaneously.

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #2, December 2006

Answers

1. $2^3 \cdot 3^2 \cdot 11$
or
$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 11$

2. 132

3. 2:48 PM

Students with correct answer in a cluster of 6 schools:

1. 30/36

2. 31/36

3. 21/36!

1. The prime factorization of 792 can be written in exponential form, with the primes listed from least to greatest as $\mathbf{2^3 \cdot 3^2 \cdot 11}$ or $\mathbf{2^3 \cdot 3^2 \cdot 11^1}$. Without exponents, we would write $\mathbf{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 11}$. We will also accept the symbol $\times$ for multiplication.

2. In general, the product of the GCF and the LCM of two numbers is equal to the product of the two numbers. We know the product of the two numbers, so if we divide by the GCF we will get the LCM: 396 ÷ 3 = **132**.

3. Gadgets take 1 hour and 24 minutes, or 60 + 24 = 84 minutes. Gizmos take 1 hour and 52 minutes, or 60 + 52 = 112 minutes. We are looking for the least common multiple of 84 and 112. The prime factorizations are $84 = 2^2 \cdot 3 \cdot 7$ and $112 = 2^4 \cdot 7$. Taking the greatest number of each prime factor, we find that the LCM is $2^4 \cdot 3 \cdot 7 = 336$. This means that after 336 minutes, a new gadget and a new gizmo will again roll off the assembly line simultaneously. This many minutes is 5 hours and 36 minutes, so we need to find the time of day that is 5 hours and 36 minutes after 9:12 AM. In military time, this would be 14:48. In regular time, it will be **2:48 PM**.

Category 4
Arithmetic
Meet #2, December 2006

1. What is 20% of $\frac{5}{8}$ of $\frac{4}{9}$ of 162?

2. Find the value of $\frac{0.8\overline{3}}{0.41\overline{6}}$. Express your answer in simplest form.

3. The repeating decimal for $\frac{1}{17}$ is $0.\overline{0588235294117647}$. What digit will appear in the twenty-third decimal place of the decimal equivalent for $1-\frac{1}{17}$?

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 4
Arithmetic
Meet #2, December 2006

Answers

1. 9

2. 2

3. 4

Students with correct answer in a cluster of 6 schools:

1. 31/36

2. 19/36!
(the most difficult of this meet)

3. 23/36

.

1. It helps to convert the percent to a fraction: 20% = $\frac{20}{100}=\frac{1}{5}$. The word "of" generally means multiply, so the value we are looking for is $\frac{1}{5}\times\frac{5}{8}\times\frac{4}{9}\times 162=\frac{162}{18}=\mathbf{9}$.

2. To convert each repeating decimal to a fraction, we use a little algebra as follows. If $x=0.8\overline{3}$, then $10x=8.3\overline{3}$. Subtracting the first equation from the second, we get $10x-x=8.3\overline{3}-0.8\overline{3}$ or $9x=7.5$. Dividing both sides of the equation by 9, we find that $x=\frac{7.5}{9}=\frac{75}{90}=\frac{5}{6}$. Similarly, if $y=0.41\overline{6}$, then $10y=4.16\overline{6}$. Again subtracting the first equation from the second, we get $10y-y=4.16\overline{6}-0.41\overline{6}$ or $9y=3.75$. Dividing both sides by 9, we find that $y=\frac{3.75}{9}=\frac{375}{900}=\frac{5}{12}$. Finally, the value of our original fraction must be $\frac{0.8\overline{3}}{0.41\overline{6}}=\frac{\frac{5}{6}}{\frac{5}{12}}=\frac{5}{6}\div\frac{5}{12}=\frac{5}{6}\times\frac{12}{5}=\mathbf{2}$.

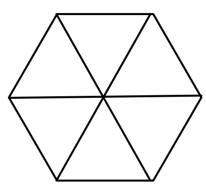
3. The repeating decimal pattern for $\frac{1}{17}$ is 16 digits long. The same 16-digit pattern occurs in the decimal equivalent of $1-\frac{1}{17}=\frac{16}{17}$, but it starts with the 9 instead of the 0. Thus we have $\frac{16}{17}=0.\overline{9411764705882352}$. The twenty-third digit in the pattern will be the same as the seventh digit in the pattern, since 23 – 16 = 7. The digit is **4**.

Category 5
Algebra
Meet #2, December 2006

1. Use the five equations below to find the value of A.

$$A^2 = 92 - B$$
$$B = C + 5$$
$$2C = \frac{D}{4}$$
$$D = 3E^4$$
$$E^5 = 32$$

2. The formula for the area of an equilateral triangle with side length s is $A = \frac{s^2\sqrt{3}}{4}$. A regular hexagon can be subdivided into six equilateral triangles as shown in the figure below. If the area of a regular hexagon is $600\sqrt{3}$ square centimeters, how many centimeters are in the side length of the hexagon?



3. The sum of three consecutive multiples of 29 is equal to the sum of four consecutive multiples of 9. If the smallest of the four multiples of 9 is 117, what is the value of the greatest of the three multiples of 29?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #2, December 2006

Answers

1. 9 (or -9)

2. 20

3. 203

Students with correct answer in a cluster of 6 schools:

1. 34/36

2. 23/36

3. 25/36

.

1. Starting with the last equation and working our way back up, we find that $E = 2$, $D = 48$, $C = 6$, $B = 11$, and $A =$ **9**.
-9 is also an acceptable answer, as $(-9)^2=81$.

2. First we will equate the area we are given with six times the formula for the area of an equilateral triangle. Then we will solve for the side length s.

$$600\sqrt{3} = 6\left(\frac{s^2\sqrt{3}}{4}\right)$$

Dividing both sides of the equation by 6 and by $\sqrt{3}$, we get $100 = \frac{s^2}{4}$. Next, we multiply both sides of the equation by 4, which gives us $400 = s^2$. Since 20^2 is 400, the side length of the hexagon must be **20** centimeters.

3. The four consecutive multiples of 9 must be 117, 126, 135, and 144. Their sum is 522. Dividing 522 by 29, we get 18. Since $5 + 6 + 7 = 18$, we can figure out that the three consecutive multiples of 29 must be $5 \times 29 = 145$, $6 \times 29 = 174$, and $7 \times 29 =$ **203**, which is the greatest.

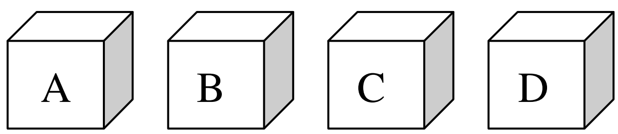
Category 6
Team Questions
Meet #2, December 2006

1. Find the sum of all the positive factors of 2006.

2. A, B, C, and D are different whole numbers. What is the greatest possible value of A if each of the statements below is true?

$$A + B < C + D, \qquad A < B, \qquad A + B + C + D < 100$$

3. Taro keeps his calculator collection in the four boxes shown below. There are three times as many calculators in box D as there are in box B. Box A has twice as many calculators as box C. Box C has five fewer calculators than box B. If there are a total of 48 calculators in the four boxes, how many calculators are there in box B?



4. What is the average of all the possible sums that use three out of five of the following numbers: 1, 2, 4, 8, and 16. Express your answer as a decimal to the nearest tenth.

5. What fraction of the factors of 96,096 are odd?

Answers
1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. __________________

6. Using the values the team obtained in questions 1 through 5, give the ***largest prime factor*** of the value of the expression below:

$$\frac{5A}{BCE} - 10D$$

Solutions to Category 6
Team Questions
Meet #2, December 2006

Answers

1. 3240

2. 24

3. 9

4. 18.6

5. $\frac{1}{6}$

6. 11

Some Incorrect Answers Seen

1. 3012, 3088
2. 23
3.
4.
5. 5/32, ½, 1/8
6.

1. The prime factorization of 2006 is $2^1 \cdot 17^1 \cdot 59^1$. The eight factors of 2006 are 1, 2, 17, 34, 59, 118, 1003, and 2006. Their sum is **3240**. This sum can also be calculated as follows: $(2+1)(17+1)(59+1) = 3 \cdot 18 \cdot 60 = 3240$.

2. Let's start with all four numbers equal to 25 and then make adjustments. We want *A* + *B* to be as close to *C* + *D* as possible, so let's have *A* + *B* = 49 and *C* + *D* = 50. Then *A* can be 24 and *B* can be 25. *C* can be 23 and *D* can be 27. The greatest possible value for *A* is **24**.

3. Suppose there are *x* calculators in box B. Then there are 3*x* calculators in box D, *x* – 5 calculators in box C, and 2(*x* – 5) calculators in box A. Since we know there are 48 calculators in all, we can write and solve an algebraic equation as shown below.

$$2(x-5)+x+(x-5)+3x=48$$
$$2x-10+x+x-5+3x=48$$
$$7x-15=48$$
$$7x=63$$
$$x=9$$

There must be **9** calculators in box B.

$$1+2+4=7$$
$$1+2+8=11$$
$$1+2+16=19$$
$$1+4+8=13$$
$$1+4+16=21$$
$$1+8+16=25$$
$$2+4+8=14$$
$$2+4+16=22$$
$$2+8+16=26$$
$$4+8+16=28$$

4. The average of the numbers 1, 2, 4, 8, and 16 is (1 + 2 + 4 + 8 + 16) ÷ 5 = 31 ÷ 5 = 6.2. If each sum includes three of the five numbers, then the average of all possible sums is 3 × 6.2 = **18.6**. The ten possible sums are shown at left to verify this result. The sum of all ten sums is 186, so the average is indeed 18.6.

5. The prime factorization of 96,096 is $2^5 \cdot 3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$. Raising each exponent by one and multiplying them, we calculate that this number has 6 × 2 × 2 × 2 × 2 = 96 factors. Any odd factor of 96,096 cannot include a factor of 2, so we must calculate how many ways we can combine the factors of 3, 7, 11, and 13 to make odd factors. Each of these primes may be used or not used to make an odd factor of 96,096, so there are 2 × 2 × 2 × 2 = 16 odd factors. That's $\frac{16}{96}$ or $\mathbf{\frac{1}{6}}$ of the factors.

6. Substituting the correct values into the expression, we get:

$$\begin{aligned}\frac{5A}{BCE}-10D &= \frac{5\times 3240}{24\times 9\times \frac{1}{6}}-10\times 18.6\\ &= \frac{16200}{36}-186\\ &= 450-186\\ &= 264 = 2^3\times 3\times 11\end{aligned}$$

The largest prime factor of 264 is **11**.